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The stability of small amplitude Rossby waves in a channel

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The breakdown of Rossby waves in a bounded system is studied for the case in which the wave amplitude is small. In a very long, laterally bounded, channel all waves are unstable via second-order resonant interactions except those of wavenumber π/L in the cross-channel direction (where L is the channel width), which are stable if their longitudinal wavenumber is greater than $0{\cdot}681\pi/L$. These waves are, however, unstable to weaker side-band interactions, so that all waves with non-zero longitudinal wavenumber are unstable. The transition from side-band to triad instability occurs where the group velocity of the basic wave is equal to the velocity of long waves.

1. Introduction

The breakdown of wave motion via nonlinear interaction with other, parasitic waves has been studied for a number of systems in recent years. In particular, Gill (1974) investigated the stability of Rossby waves on an unbounded β-plane and showed that (in an inviscid fluid) these waves are unstable for all wavenumbers. The nature of the instability depends on the size of the parameter $\epsilon = U/\beta L^2$ (where U and L are typical velocity and length scales of the wave, respectively), the ratio of relative to planetary vorticity gradients. If ϵ is small, nonlinear effects are weak, and the instability may proceed only via resonant interactions; for the highly nonlinear case (ϵ large), breakdown takes the form of a Rayleigh inflexion-point instability.

The introduction of lateral boundaries has a stabilizing effect on such motion. The role of the boundaries is somewhat different in the cases of large or small ϵ, but it depends crucially on the conservation properties of the system, in the form of the criterion of Fjørtoft (1953) that energy must be transferred simultaneously to both longer and shorter wavelengths (in such a way that both kinetic energy and enstrophy are conserved). In the highly nonlinear case (ϵ large) this constraint inhibits instability of large-scale waves since such waves cannot transfer energy to yet larger scales. Thus Hoskins (1973) found that a large amplitude Rossby wave of gravest meridional‡ structure (i.e. one half-wavelength across

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‡ For simplicity (and by analogy with geophysical systems), we refer to the directions along and across the channel as 'zonal' and 'meridional' respectively.

the channel) is unstable to zonal flow perturbations only if its zonal wavenumber k is greater than twice its meridional wavenumber l, while Baines (1976) showed that a planetary wave on a sphere is unstable, provided that its amplitude is large enough, if its total wavenumber is greater than 2.

This paper is concerned with the stability properties when ϵ is small. In this case, the additional requirement of resonance must be met. The boundaries restrict the possible meridional wavenumbers to a discrete set of values. If the channel is long enough for the zonal wavenumber to be regarded as continuous, then at least one resonant triad can be found for any value of the wavenumber (k, l) of the initial wave, but not all of these correspond to instabilities (for which the triad components must be such that Fjørtoft's criterion is satisfied). It is found that a wave of gravest meridional structure is unstable via triad interactions if the ratio k/l of its zonal and meridional wavenumbers is greater than a critical value $\eta_c = 0{\cdot}681$; waves of shorter meridional wavelength are unstable for all non-zero k.

The transition at $k/l = \eta_c$ corresponds to equality between the group velocity of the basic wave (k, l) and that of a long wave of meridional wavenumber $2l$. McIntyre (1973) demonstrated the existence of this resonance in the corresponding internal wave problem and Grimshaw (1975) showed that, at the resonance, interaction takes place on a time scale formally of order $\epsilon^{-\frac{4}{3}}$, while for $k/l < \eta_c$ a wave is unstable to side-band interactions (i.e. modulation of the basic wave via interaction with a long wave) on a time scale ϵ^{-2}. Analogous results are obtained in this case, with the solution in the regime of long-wave resonance providing continuity between the side-band and triad instabilities.

2. The interaction equation

Consider two-dimensional motion of fluid on a β-plane bounded meridionally by rigid walls at $y' = 0, \pi L$ and unbounded in the x' (zonal) direction. The governing equation is the non-divergent barotropic vorticity equation

$$\frac{\partial}{\partial t}(\nabla'^2\psi') + \beta\frac{\partial\psi'}{\partial x'} + \frac{\partial(\psi', \nabla'^2\psi')}{\partial(x', y')} = 0, \tag{1}$$

where t' is time, ψ' the stream function such that the velocity components in the (x', y') directions are

$$(u', v') = (-\partial\psi'/\partial y', \partial\psi'/\partial x'),$$

and $\nabla'^2 \equiv \partial^2/\partial x'^2 + \partial^2/\partial y'^2$. Then, choosing a typical velocity scale U, we define the dimensionless variables

$$x = x'/L, \quad y = y'/L, \quad t = \beta L t', \quad \psi = \psi'/UL,$$

upon which (1) becomes

$$\frac{\partial}{\partial t}(\nabla^2\psi) + \frac{\partial\psi}{\partial x} = -\epsilon\frac{\partial(\psi, \nabla^2\psi)}{\partial(x, y)}, \tag{2}$$

where $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$ and $\epsilon = U/\beta L^2$. The parameter ϵ is a measure of the nonlinearity (Gill 1974), and this paper is concerned with the case $\epsilon \ll 1$.

The meridional boundary conditions are satisfied by a set of waves of the form

$$\psi = \sum_n \psi_n = \sum_n a_n(t) \exp(ik_n x) \sin l_n y \tag{3}$$

(where l_n is a positive integer), provided that $k_n \neq 0$. If we introduce the notation that if $\mathbf{k}_n = (k_n, l_n)$ then $\mathbf{k}_n^* = (-k_n, l_n)$, then, for real ψ, $a_m = a_n^*$, where $\mathbf{k}_m = \mathbf{k}_n^*$.

Several time scales arise in the problem in addition to the propagation time scale t; the triad interaction takes place on the time scale T_1 (or order ϵ^{-1}), while that of the weaker side-band interaction, T_2, is of order ϵ^{-2} (Newell 1969). Further, we introduce a long space scale X, corresponding to a bandwidth of the resonant interactions of order ϵ in wavenumber space (see Bretherton (1964) and below), so that

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial T_1} + \epsilon^2 \frac{\partial}{\partial T_2}$$

and

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \epsilon \frac{\partial}{\partial X}.$$

Then in (3) we write

$$a_n = b_n(X, T_1, T_2) \exp(-i\omega_n t), \tag{4}$$

so that (2) becomes the interaction equation

$$\mathscr{F}(b_n) = \epsilon \sum_p \sum_q \mathscr{N}_n(b_p, b_q)\, \delta(k_p + k_q - k_n) \exp[i(\omega_n - \omega_p - \omega_q)t], \tag{5}$$

where the operators $\mathscr{F}$ and $\mathscr{N}_n$ may be expanded as

$$\begin{aligned}\mathscr{F}(b_n) = {} & f_n b_n + i\epsilon \left\{ \left(\frac{\partial f}{\partial \omega}\right)_n \frac{\partial}{\partial T_1} - \left(\frac{\partial f}{\partial k}\right)_n \frac{\partial}{\partial X} \right\} b_n \\ & + \epsilon^2 \left\{ i \left(\frac{\partial f}{\partial \omega}\right)_n \frac{\partial}{\partial T_2} - \frac{1}{2} \left[\left(\frac{\partial^2 f}{\partial \omega^2}\right)_n \frac{\partial^2}{\partial T_1^2} - 2 \left(\frac{\partial^2 f}{\partial \omega \partial k}\right)_n \frac{\partial^2}{\partial X \partial T_1} \right.\right. \\ & \left.\left. + \left(\frac{\partial^2 f}{\partial k^2}\right)_n \frac{\partial^2}{\partial X^2} \right] \right\} b_n + O(\epsilon^3)\end{aligned} \tag{6}$$

and

$$\begin{aligned}\mathscr{N}_n(b_p, b_q) = {} & \gamma(\mathbf{k}_n, \mathbf{k}_p, \mathbf{k}_q)\, b_p b_q \\ & - i\epsilon\{\mu(\mathbf{k}_n, \mathbf{k}_p, \mathbf{k}_q)\, b_q\, \partial b_p/\partial X + \nu(\mathbf{k}_n, \mathbf{k}_p, \mathbf{k}_q)\, b_p\, \partial b_q/\partial X\} + O(\epsilon^2),\end{aligned} \tag{7}$$

and where

$$f_n \equiv f(\mathbf{k}_n, \omega_n) = \omega_n(k_n^2 + l_n^2) + k_n. \tag{8}$$

The interaction coefficients γ, μ and ν are given by

$$\begin{aligned}\gamma(\mathbf{k}_n, \mathbf{k}_p, \mathbf{k}_q) = {} & \tfrac{1}{2}(k_q^2 + l_q^2)\{(k_q l_p - k_p l_q)[\delta(l_p + l_q - l_n) - \delta(l_p + l_q + l_n)] \\ & + (k_p l_q + k_q l_p)[\delta(l_p - l_q + l_n) - \delta(l_p - l_q - l_n)]\},\end{aligned} \tag{9}$$

$$\left.\begin{aligned}\mu(\mathbf{k}_n, \mathbf{k}_p, \mathbf{k}_q) &= \partial\gamma(\mathbf{k}_n, \mathbf{k}_p, \mathbf{k}_q)/\partial k_p, \\ \nu(\mathbf{k}_n, \mathbf{k}_p, \mathbf{k}_q) &= \partial\gamma(\mathbf{k}_n, \mathbf{k}_p, \mathbf{k}_q)/\partial k_q.\end{aligned}\right\} \tag{10}$$

We note that

$$\left.\begin{aligned}\gamma(\mathbf{k}_n, \mathbf{k}_p, \mathbf{k}_q^*) &= -\gamma(\mathbf{k}_n, \mathbf{k}_p^*, \mathbf{k}_q), \\ \mu(\mathbf{k}_n, \mathbf{k}_p, \mathbf{k}_q^*) &= \mu(\mathbf{k}_n, \mathbf{k}_p^*, \mathbf{k}_q), \\ \nu(\mathbf{k}_n, \mathbf{k}_p, \mathbf{k}_q^*) &= \nu(\mathbf{k}_n, \mathbf{k}_p^*, \mathbf{k}_q).\end{aligned}\right\} \tag{11}$$

3. Stability of waves to resonant triads

Consider a triad of interacting waves with wavenumbers $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$, where

$$k_1 + k_2 + k_3 = 0, \quad l_1 \pm l_2 \pm l_3 = 0. \tag{12}$$

Evolution takes place on the time scale T_1, and we write

$$\psi_n = \{b_n(X, T_1)\exp[i(k_n x - \omega_n t)] + \text{c.c.}\}\sin l_n y, \tag{13}$$

where c.c. denotes 'complex conjugate'. If the triad is resonant, i.e.

$$\omega_1 + \omega_2 + \omega_3 = 0 \tag{14}$$

(it will be shown below that such non-trivial solutions can be found), then all other wave components are negligible to the order of interest.

To leading order the interaction equation (5) gives the dispersion relation

$$f(\mathbf{k}_n, \omega_n) = 0, \tag{15}$$

while to order ϵ we find

$$\left(\frac{\partial}{\partial T_1} + C_1\frac{\partial}{\partial X}\right)b_1 = -i\alpha(\mathbf{k}_1, \mathbf{k}_2^*, \mathbf{k}_3^*)\, b_2^* b_3^* \tag{16}$$

together with two similar equations for b_2 and b_3, where the group velocity is

$$C_n = -\left(\frac{\partial f}{\partial k}\right)_n \bigg/ \left(\frac{\partial f}{\partial \omega}\right)_n \tag{17}$$

and the interaction coefficient

$$\alpha(\mathbf{k}_n, \mathbf{k}_p, \mathbf{k}_q) = \{\gamma(\mathbf{k}_n, \mathbf{k}_p, \mathbf{k}_q) + \gamma(\mathbf{k}_n, \mathbf{k}_q, \mathbf{k}_p)\} \bigg/ \left(\frac{\partial f}{\partial \omega}\right)_n. \tag{18}$$

It follows from (9) and (18) that $\alpha(\mathbf{k}_n, \mathbf{k}_p, \mathbf{k}_p) = 0$, so that a single steady wave, described by $b_1 = A$ and $b_2 = b_3 = 0$, is a solution to (16), and we investigate the stability of such a state by adding perturbations $b_2 = z_2$ and $b_3 = z_3$, where $|z_n| \ll |A|$ (but, for consistency, $|z_n| \gg \epsilon$). Linearizing, we obtain the equations

$$\left.\begin{aligned} (\partial/\partial T_1 + C_2\,\partial/\partial X)\, z_2 &= -i\alpha(\mathbf{k}_2, \mathbf{k}_3^*, \mathbf{k}_1^*)\, A^* z_3^* \\ \text{and} \qquad (\partial/\partial T_1 + C_3\,\partial/\partial X)\, z_3 &= -i\alpha(\mathbf{k}_3, \mathbf{k}_1^*, \mathbf{k}_2^*)\, A^* z_2^*, \end{aligned}\right\} \tag{19}$$

which have solutions of the form

$$z_2 \propto z_3^* \propto \exp\{i(KX - \Omega T_1)\}$$

if $$\Omega = \tfrac{1}{2}K(C_2 + C_3) \pm \{\tfrac{1}{4}K^2(C_2 - C_3)^2 - \alpha(\mathbf{k}_2, \mathbf{k}_3^*, \mathbf{k}_1^*)\,\alpha(\mathbf{k}_3, \mathbf{k}_1^*, \mathbf{k}_2^*)|A|^2\}^{\frac{1}{2}}. \tag{20}$$

Hence the wave is unstable to such a perturbation if the condition

$$K^2(C_2 - C_3)^2 < 4\alpha(\mathbf{k}_2, \mathbf{k}_3^*, \mathbf{k}_1^*)\,\alpha(\mathbf{k}_3, \mathbf{k}_1^*, \mathbf{k}_2^*)|A|^2 \tag{21}$$

is satisfied. Clearly, no matter now small K^2, instability may occur only if

$$\alpha(\mathbf{k}_3, \mathbf{k}_1^*, \mathbf{k}_2^*)\,\alpha(\mathbf{k}_2, \mathbf{k}_3^*, \mathbf{k}_1^*) > 0. \tag{22}$$

Inequality (22) is a criterion deriving from the physical constraints on the system, which, in the present case, are the conservation of kinetic energy and enstrophy. Substituting for the interaction coefficients, it may be rewritten (Hoskins 1973; Gill 1974) as

$$\{(k_1^2+l_1^2)-(k_2^2+l_2^2)\}\{(k_1^2+l_1^2)-(k_3^2+l_3^2)\} < 0,$$

which is a statement of the anti-cascade theorem of Fjørtoft (1953): energy transfer to higher wavenumbers must be accompanied by simultaneous transfer to lower wavenumbers. One may also use the dispersion relation to show that (22) implies that $\omega_2\omega_3 > 0$, so that, from (14), $|\omega_1| > \max(|\omega_2|, |\omega_3|)$. This is Hasselmann's (1967) criterion that a wave is unstable via interactions within a resonant triad only if it is the component of the triad of highest frequency.

If inequality (22) is satisfied, then instability will proceed if the triad is close enough to resonance. In order that energy may be exchanged coherently over the long time scale T_1 of the interaction, the sum of the (finite amplitude) frequencies of the waves must be zero at least to order ϵ. If K is non-zero, the triad is not exactly resonant in a linear sense, the wave frequencies being ω_1, $\omega(k_2+\epsilon K, l_2) = \omega_2+\epsilon KC_2+O(\epsilon^2)$ and $\omega(k_3-\epsilon K, l_3) = \omega_3-\epsilon KC_3+O(\epsilon^2)$, so that, using the resonance condition (14), their sum is $\epsilon K(C_2-C_3)+O(\epsilon^2)$. Thus condition (21) merely expresses the requirement that finite amplitude effects be large enough to overcome this dispersion, and defines a resonant spectral bandwidth Δ_{123} for the interaction, such that, if $K = \pm\Delta_{123}$, inequality (21) becomes an equality.

If the velocity scale U is chosen such that $A = 1$, then the maximum growth rate, attained when $K = 0$, is $\epsilon\sigma$, where

$$\sigma = (-i\Omega)_{\max} = \{\alpha(\mathbf{k}_2, \mathbf{k}_3^*, \mathbf{k}_1^*)\,\alpha(\mathbf{k}_3, \mathbf{k}_1^*, \mathbf{k}_2^*)\}^{\frac{1}{2}}. \tag{23}$$

4. Results for Rossby waves in a channel

The quadratic instability problem for a small amplitude Rossby wave in a channel has been reduced to that of finding two other members of a resonant triad whose wavenumbers are such that energy flow from the initial wave into the two parasitic waves is consistent with the physical constraints on the motion.

In an unbounded fluid, resonant triads can always be found (Longuet-Higgins & Gill 1967) but in the present case the triads which simultaneously satisfy the kinematic resonance conditions (12) and (14) must also satisfy the constraint that the meridional wavenumbers of the triad components be integers. Calculations and experiments involving internal wave stability in a channel (Martin, Simmons & Wunsch 1972) and in a system bounded in both directions (McEwan, Mander & Smith 1972) show that resonant breakdown invariably occurs (for small enough viscosity). However, unlike Rossby waves, internal wave motions are not constrained by enstrophy conservation and instability may, as an extreme example, proceed by transfer of energy solely to much smaller scales (McEwan & Robinson 1975) for which the constraints of boundedness are less important (Plumb 1977).

In practice it is straightforward to locate the resonant triads that include a

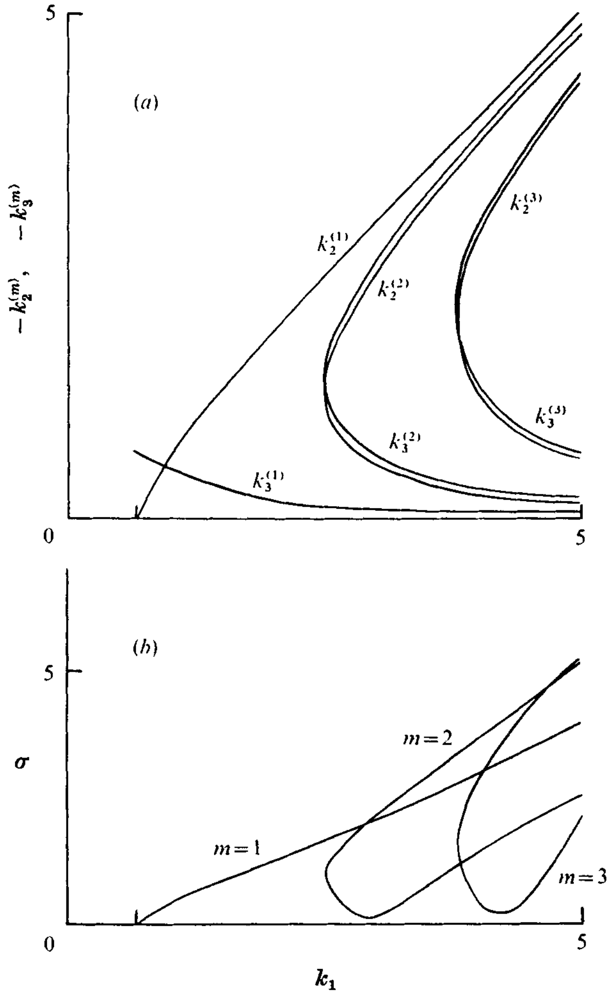

FIGURE 1. (*a*) Zonal wavenumbers $k_2^{(m)}$, $k_3^{(m)}$ of the mth growing mode perturbation to an initial wave $\mathbf{k}_1 = (k_1, 1)$. Meridional wavenumbers: $l_2^{(1)} = 2$, $l_3^{(1)} = 1$; $l_2^{(2)} = 3$, $l_3^{(2)} = 2$; $l_2^{(3)} = 4$, $l_3^{(3)} = 3$. (*b*) Growth rates of the mth mode for exact resonance, i.e. $K = 0$ with $|A| = 1$.

given wave $\mathbf{k}_1$ by scanning (numerically) $\mathbf{k}_2$ space (for $l_1 \neq l_2$, there are two corresponding values of $\mathbf{k}_3$, with $l_3 = |l_1 \pm l_2|$) and then using the dispersion relation to evaluate $\omega_1 + \omega_2 + \omega_3$ as a function of $\mathbf{k}_2$. The zeros of this function give the resonant triads. It transpires that, for any $\mathbf{k}_1$, at least one resonant triad can be found, but it is not always possible to satisfy condition (22). The perturbation wavenumbers $\mathbf{k}_2$ and $\mathbf{k}_3$ to which wave 1 is unstable, satisfying (12), (14) and (22), are plotted for the case $l_1 = 1$ in figure 1(*a*). If $k_1 < 0{\cdot}681$ no unstable modes exist, whereas at least one such mode can be found for all $k_1 > 0{\cdot}681$. The growth rates σ (for $|A| = 1$) are shown in figure 1(*b*); as k_1 increases, the most rapidly growing perturbations are those with increasingly large meridional wavenumbers. Figure 2 shows the unstable modes and growth rates for $l_1 = 2$;

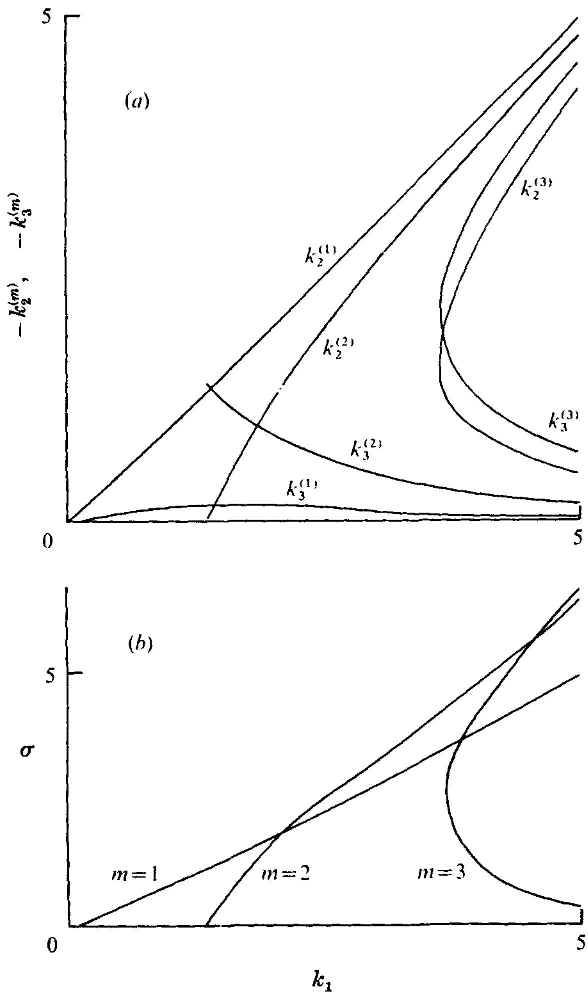

FIGURE 2. As figure 1, but with $\mathbf{k}_1 = (k_1, 2)$. Meridional wavenumbers: $l_2^{(1)} = 3$, $l_3^{(1)} = 1$; $l_2^{(2)} = 4$, $l_3^{(2)} = 2$; $l_2^{(3)} = 3$, $l_3^{(3)} = 5$.

in this case there is at least one perturbation mode to which the wave is unstable for any non-zero k_1. Calculations with $2 < l_1 \leqslant 5$ (not presented here) show that this is true in these cases also.

Note that all the growing perturbations are such that energy is transferred to longer zonal wavelengths only. This corresponds to the tendency for weakly interacting eddies on a β-plane to become east–west orientated in the study of Rhines (1975) and is consistent with the anti-cascade theorem; the meridional scale of the motion is decreasing (at least one growing mode has $l \geqslant 2$), so that, in order that both kinetic energy and enstrophy be conserved, the zonal scale must increase. Indeed, the generality of this behaviour follows from Hasselmann's criterion. If $k_1 > 0$, then the dispersion relation

$$\omega_n = -k_n/(k_n^2 + l_n^2) \tag{24}$$

gives $\omega_1 < 0$. Hence, with the sign convention adopted in (12) and (14), ω_2 and ω_3 must both be positive, since $|\omega_2|$ and $|\omega_3|$ must both be less than $|\omega_1|$ for instability. It then follows directly from (24) that k_2 and k_3 are negative and hence that $|k_2| < |k_1|$ and $|k_3| < |k_1|$. Thus resonant breakdown of a Rossby wave *always* involves transfer of energy to zonally longer waves only, whether or not the fluid is bounded.

The nature of the transition at $k_1 = 0{\cdot}681$ with $l_1 = 1$ is important in the interpretation of the results of the following sections. This mode (labelled $m = 1$ in figure 1) has $\mathbf{k}_1 = (k_1, l)$, $\mathbf{k}_2 = (k_2, 2l)$ and $\mathbf{k}_3 = (k_3, l)$. In the region of the transition, $k_2 = -\delta k$ and $k_3 = \delta k - k_1$, where $\delta k \ll k_1$. Hence $\omega_2 \simeq C_2\,\delta k$ and $\omega_3 \simeq -\omega_1 + C_1\,\delta k$, where $C_1 = C(k_1, l)$ and $C_2 = C(0, 2l)$ are the group velocities of the basic wave and of long waves respectively. Then $\omega_1 + \omega_2 + \omega_3 \simeq (C_1 - C_2)\,\delta k$ and so the requirement of resonance is that $C_1 = C_2$ at the transition. Since

$$C_1 - C_2 = (3l^2 - 6k_1^2 l^2 - k_1^4)/4l^2(k_1^2 + l^2)^2 \tag{25}$$

the transition occurs where

$$k_1/l_1 = \eta_c = 0{\cdot}681, \tag{26}$$

in agreement with the results in figure 1. With $l_1 = 2$, the transition for mode 2 in figure 2 is found at $k_1 = 2\eta_c$; in this case, however, the wave $(k_1, 2)$ is unstable to perturbations of mode 1 when $k_1 < 2\eta_c$, so that the mode 2 transition is less significant.

In the region of this transition, say $k_1 = \eta_c l + \Delta$, where $\Delta \ll \eta_c l$, with $k_2 = -\delta k$ we have

$$\omega_1 + \omega_2 + \omega_3 = \delta k\,(\Delta - \tfrac{1}{2}\delta k)\,(\partial^2\omega/\partial k^2)_{1c},$$

where we have used $(\partial^2\omega/\partial k^2)_{2c} = 0$ and where the subscript c refers to values at the long-wave resonance, i.e. $k_1 = \eta_c l, k_2 = 0$. If follows that

$$\delta k = 2\Delta. \tag{27}$$

Now, since $|\mathbf{k}_2| > |\mathbf{k}_1|$ at the transition, the inequality $|\mathbf{k}_3| < |\mathbf{k}_1|$ must be satisfied for instability. Hence $\delta k > 0$ and (27) shows that Δ must be positive, i.e. the wave is unstable via this interaction only for $k_1 > \eta_c l$, as the calculations show.

To conclude this section we note that validity of the triad concept requires that interaction other than that indicated by the sign convention in (12) and (14), in triads such as $(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_1 - \mathbf{k}_2)$, be non-resonant. However, if δk, $\Delta \leqslant O(\epsilon)$ the mode $\mathbf{k}_4 = \mathbf{k}_1 - \mathbf{k}_2 = (\eta_c l + \Delta + \delta k, l)$ forms another triad $\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_4 = 0$ for which $\omega_1 - \omega_2 - \omega_4 \leqslant O(\epsilon)$. Hence, from the arguments of the previous section, we may not neglect the contribution of this extra component and the triad analysis breaks down unless $\Delta > O(\epsilon)$.

5. Instability via resonant side bands

In addition to triad interactions, the requirements of resonance may be met by quartet or higher-order resonances, in which case interaction takes place on a time scale ϵ^{-2} or longer. Another class of interaction is the side-band resonance, in which the basic wave interacts with and transfers energy into waves of slightly different wavenumber and long waves; the process may be regarded as one of spectral broadening. The mechanism was demonstrated for water waves by Benjamin & Feir (1967), while Newell (1969) showed that such interaction may occur in unbounded Rossby waves. This interaction causes instability of long internal waves in a bounded geometry analogous to that of the present problem (Grimshaw 1975).

The waves participating in the interaction are waves with wavenumbers, say, $\mathbf{k}_1 = (k, l)$ and $\mathbf{k}_1^{\pm} = (k \pm \delta k, l)$ and a long wave with $\mathbf{k}_2 = (\delta k, 2l)$. The structure is not that of a quartet resonance, but of two almost resonant triads with the common members $\mathbf{k}_1$ and $\mathbf{k}_2$; the two triads (1, 2, $1^{\pm}$) have

$$k_1 \pm k_2 - k_1^{\pm} = 0$$

and
$$\omega_1 \pm \omega_2 - \omega_1^{\pm} = \pm (C_2 - C_1)\,\delta k + O(\delta k^2).$$

Hence for δk small enough ($\delta k \leqslant O(\epsilon)$) both triads become resonant. (Interactions of modes 1 and $1^{\pm}$, driving the harmonic, are non-resonant and therefore negligible.) Note that, since $\alpha(\mathbf{k}_2, \mathbf{k}_1, \mathbf{k}_1) = 0$, the interaction coefficients $\alpha(\mathbf{k}_2, \mathbf{k}_1, \mathbf{k}_1^{\pm})$ governing this process are proportional to δk and so, with $\delta k = O(\epsilon)$, the growth rates are at most $O(\epsilon^2)$ (Newell 1969).

With δk of this order it is sufficient to consider the interaction between the two 'carrier' waves $\mathbf{k}_1 = (k, l)$ and $\mathbf{k}_2 = (0, 2l)$ and to introduce the side-band and long-wave structure as a function of the slow variable X. Because of this weakness of the interaction, it becomes necessary to include the effect of the non-resonant component $(k, 3l)$; all other wave components are negligible.

We write

$$\left.\begin{aligned} \psi_1 &= \{b_1(X, T_1, T_2)\, e^{i(kx-\omega t)} + \text{c.c.}\}\sin ly, \\ \psi_2 &= \{b_2(X, T_1, T_2) + \text{c.c.}\}\sin 2ly, \\ \psi_3 &= \{b_3(X, T_1, T_2)\, e^{i(kx-\omega t)} + \text{c.c.}\}\sin 3ly, \end{aligned}\right\} \tag{28}$$

where $\omega = \omega(k, l)$, and perturb about a mean state, such that

$$b_1 = A + z_1, \quad b_2 = z_2, \quad b_3 = z_3,$$

where z_n is a small perturbation (but $|z_n| \gg \epsilon$), which is expanded as

$$z_n = z_n^{(0)}(X, T_1, T_2) + \epsilon z_n^{(1)}(X, T_1, T_2) + O(\epsilon^2). \tag{29}$$

To order ϵ^0, the interaction equation (5) gives

$$z_3^{(0)} = 0.$$

The order-ϵ equations become, after some simplification,

$$(\partial/\partial T_1 + C_1\, \partial/\partial X)\, z_1^{(0)} = -i\alpha(\mathbf{k}_1, \mathbf{k}_1, \mathbf{k}_2)\,(z_2^{(0)} + z_2^{(0)*})\,A, \tag{30a}$$

$$(\partial/\partial T_1 + C_2\, \partial/\partial X)\, z_2^{(0)} = 0, \tag{30b}$$

$$f(\mathbf{k}_3, \omega)\, z_3^{(1)} = (\partial f/\partial \omega)_3\, \alpha(\mathbf{k}_3, \mathbf{k}_1, \mathbf{k}_2)\,(z_2^{(0)} + z_2^{(0)*})\,A. \tag{30c}$$

Equations (30) have the solutions

$$\text{(i)} \quad z_2^{(0)} = M_2(T_2)\exp[iK(X-C_2T_1)] + N_2(T_2)\exp[-iK(X-C_2T_1)], \tag{31a}$$

$$z_1^{(0)} = -\frac{\alpha(\mathbf{k}_1,\mathbf{k}_1,\mathbf{k}_2)A}{K(C_1-C_2)}\{(M_2+N_2^*)\exp[iK(X-C_2T_1)] - (M_2^*+N_2)\exp[-iK(X-C_2T_1)]\}, \tag{31b}$$

with
$$z_3^{(1)} = \frac{(\partial f/\partial\omega)_3}{f(\mathbf{k}_3,\omega)}\alpha(\mathbf{k}_3,\mathbf{k}_1,\mathbf{k}_2)A(z_2^{(0)}+z_2^{(0)*}), \tag{31c}$$

and
$$\text{(ii)} \quad z_1^{(0)} = M_1(T_2)\exp[iK(X-C_1T_1)] + N_1(T_2)\exp[-iK(X-C_1T_1)], \tag{32a}$$

$$z_2^{(0)} = z_3^{(1)} = 0. \tag{32b, c}$$

So, to leading order, the solution consists of (i) two wave packets propagating at the group velocity of component 2 or (ii) a wave packet of component 1 propagating at its own group velocity; the nonlinear feedback is not yet apparent.

To order ϵ^2, the two equations describing the evolution of components 1 and 2 on the time scale T_2 are

$$\left\{\frac{\partial}{\partial T_2} + \frac{i}{(\partial f/\partial\omega)_1}\left[\frac{1}{2}\left(\frac{\partial^2 f}{\partial\omega^2}\right)_1\frac{\partial^2}{\partial T_1^2} - \left(\frac{\partial^2 f}{\partial\omega\,\partial k}\right)_1\frac{\partial^2}{\partial X\partial T_1} + \frac{1}{2}\left(\frac{\partial^2 f}{\partial k^2}\right)_1\frac{\partial^2}{\partial X^2}\right]\right\} \times z_1^{(0)} + \left\{\frac{\partial}{\partial T_1} + C_1\frac{\partial}{\partial X}\right\} z_1^{(1)}$$
$$= -i\alpha(\mathbf{k}_1,\mathbf{k}_1,\mathbf{k}_2)A(z_2^{(1)}+z_2^{(1)*}) - \beta(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1)A\frac{\partial}{\partial X}(z_2^{(0)}+z_2^{(0)*}) \tag{33a}$$

and

$$\left\{\frac{\partial}{\partial T_2} + \frac{i}{(\partial f/\partial\omega)_2}\left[\frac{1}{2}\left(\frac{\partial^2 f}{\partial\omega^2}\right)_2\frac{\partial^2}{\partial T_1^2} - \left(\frac{\partial^2 f}{\partial\omega\,\partial k}\right)_2\frac{\partial^2}{\partial X\partial T_1} + \frac{1}{2}\left(\frac{\partial^2 f}{\partial k^2}\right)_2\frac{\partial^2}{\partial X^2}\right]\right\} \times z_2^{(0)} + \left\{\frac{\partial}{\partial T_1} + C_1\frac{\partial}{\partial X}\right\} z_2^{(1)}$$
$$= -i\alpha(\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_3^*)(Az_3^{(1)*} - A^*z_3^{(1)}) - \beta(\mathbf{k}_2,\mathbf{k}_1^*,\mathbf{k}_1)\left(A\frac{\partial z_1^{(0)*}}{\partial X} + A^*\frac{\partial z_1^{(0)}}{\partial X}\right), \tag{33b}$$

where
$$\beta(\mathbf{k}_n,\mathbf{k}_p,\mathbf{k}_q) = \{\mu(\mathbf{k}_n,\mathbf{k}_p,\mathbf{k}_q) + \nu(\mathbf{k}_n,\mathbf{k}_q,\mathbf{k}_p)\}/(\partial f/\partial\omega)_n. \tag{34}$$

Now consider the evolution of mode (i) on the time scale T_2. Substituting from (28) into (30b), and suppressing secularities on the time scale T_1, we find

$$\left\{\frac{\partial}{\partial T_2} + i\lambda_2 K^2\right\} M_2 = \left\{\frac{\partial}{\partial T_2} + i\lambda_2 K^2\right\} N_2 = 0, \tag{35}$$

where
$$\lambda_n = \frac{-1}{(\partial f/\partial\omega)_n}\left\{\frac{1}{2}\left(\frac{\partial^2 f}{\partial\omega^2}\right)_n C_n^2 + \left(\frac{\partial^2 f}{\partial\omega\,\partial k}\right)_n C_n + \frac{1}{2}\left(\frac{\partial^2 f}{\partial k^2}\right)_n\right\}. \tag{36}$$

The wave component 2 is unaffected by the nonlinearity (whose only role is to force non-zero components 1 and 3), and the initial wave is therefore stable to such a perturbation.

For mode (ii), (30*b*) gives

$$z_2^{(1)} = \frac{-\beta(\mathbf{k}_2, \mathbf{k}_1^*, \mathbf{k}_1)}{(C_2 - C_1)} \{(AN_1^* + A^*M_1)\exp[iK(X - C_1 T_1)] + (AM_1^* + A^*N_1)\exp[-iK(X - C_1 T_1)]\}. \tag{37}$$

Then suppression of secularities in (30*a*) leads to the two equations

$$\left.\begin{aligned} \left\{\frac{\partial}{\partial T_2} + i\lambda_1 K^2\right\} M_1 &= \frac{2i\alpha(\mathbf{k}_1, \mathbf{k}_1, \mathbf{k}_2)\beta(\mathbf{k}_2, \mathbf{k}_1^*, \mathbf{k}_1)}{(C_2 - C_1)} \{A^2 N_1^* + |A|^2 M_1\} \\ \text{and} \quad \left\{\frac{\partial}{\partial T_2} + i\lambda_1 K^2\right\} N_1 &= \frac{2i\alpha(\mathbf{k}_1, \mathbf{k}_1, \mathbf{k}_2)\beta(\mathbf{k}_2, \mathbf{k}_1^*, \mathbf{k}_1)}{(C_2 - C_1)} \{A^2 M_1^* + |A|^2 N_1\}. \end{aligned}\right\} \tag{38}$$

Looking for solutions $M_1 \propto N_1^* \propto \exp(i\Omega T_2)$, we find

$$\Omega^2 = \lambda_1^2 K^2 \left\{K^2 - \frac{4\alpha(\mathbf{k}_1, \mathbf{k}_1, \mathbf{k}_2)\beta(\mathbf{k}_2, \mathbf{k}_1^*, \mathbf{k}_1)|A|^2}{(C_2 - C_1)\lambda_1}\right\}, \tag{39}$$

so that the initial wave $\psi = A\cos(kx - \omega t)\sin ly$ is unstable to side-band perturbations if

$$\alpha(\mathbf{k}_1, \mathbf{k}_1, \mathbf{k}_2)\beta(\mathbf{k}_2, \mathbf{k}_1^*, \mathbf{k}_1)/\lambda_1(C_2 - C_1) > 0 \tag{40}$$

and

$$K^2 < |4\alpha(\mathbf{k}_1, \mathbf{k}_1, \mathbf{k}_2)\beta(\mathbf{k}_2, \mathbf{k}_1^*, \mathbf{k}_1)/\lambda_1(C_2 - C_1)|\,|A|^2. \tag{41}$$

Condition (40) gives

$$4k^2l^2(k^2 + l^2)^4/(k^4 + 6k^2l^2 - 3l^4) > 0,$$

i.e. $k/l < \eta_c$: waves of zonal wavelength longer then $2\pi/\eta_c l$ are unstable. The maximum growth rate $\epsilon^2\sigma$ is attained when

$$K^2 = \frac{2\alpha(\mathbf{k}_1, \mathbf{k}_1, \mathbf{k}_2)\beta(\mathbf{k}_2, \mathbf{k}_1^*, \mathbf{k}_1)}{(C_2 - C_1)\lambda_1}|A|^2$$

and σ is given, with $|A| = 1$, by

$$\sigma = (-i\Omega)_{\max} = |2\alpha(\mathbf{k}_1, \mathbf{k}_1, \mathbf{k}_2)\beta(\mathbf{k}_2, \mathbf{k}_1^*, \mathbf{k}_1)/(C_2 - C_1)| \tag{42}$$

for $k/l < \eta_c$; as a function of wavenumber, it has the form

$$\sigma = k^2l^2[3 - (k/l)^2][1 + (k/l)^2]/[3 - 6(k/l)^2 - (k/l)^4].$$

The quantity σ/k^2l^2 increases indefinitely as $k/l \to \eta_c$; the theory becomes invalid close to the long-wave resonance when $|k - \eta_c l| \leqslant O(\epsilon)$ and it is necessary to consider this case separately.

6. Long-wave resonance

The preceding analyses of resonant-triad and side-band instability both break down when $k \simeq \eta_c l$, i.e. when the group velocity of the basic wave is equal to the long-wave (group and phase) velocity. This resonance has been considered by

Grimshaw (1975) for the case of internal waves; he showed that, retaining ϵ as the amplitude scale, the correct expansions for the space and time derivatives are

$$\left.\begin{aligned} &\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \epsilon^{\frac{2}{3}} \frac{\partial}{\partial X'}, \\ \text{and} \qquad &\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \epsilon^{\frac{2}{3}} \frac{\partial}{\partial T_1'} + \epsilon^{\frac{4}{3}} \frac{\partial}{\partial T_2'}. \end{aligned}\right\} \tag{43}$$

The perturbations are expanded as

$$\left.\begin{aligned} z_1 &= z_1^{(0)} + \epsilon^{\frac{2}{3}} z_1^{(1)} + \dots, \\ z_2 &= \epsilon^{\frac{1}{3}} [z_2^{(0)} + \epsilon^{\frac{2}{3}} z_2^{(1)} + \dots]. \end{aligned}\right\} \tag{44}$$

(Mode 3 as defined in §5 plays no part in this interaction, which is stronger than the side-band instability.) Consistent with the space scale in (43), the region of parameter space under consideration is that close to the long-wave resonance, for which

$$k - \eta_c l = \Gamma \epsilon^{\frac{2}{3}}, \tag{45}$$

where Γ is of order unity, so that

$$C_1(k, l) \simeq C_2(0, 2l) + 2\lambda_1 \Gamma \epsilon^{\frac{2}{3}}, \tag{46}$$

where λ_1 is defined by (36).

Now the expansion of the operator $\mathscr{F}$ in (6) goes through in the same way with the new scaling (43); the leading-order equations, using (46), are

$$\left.\begin{aligned} (\partial/\partial T_1' + C_2 \partial/\partial X') z_1^{(0)} = 0, \\ (\partial/\partial T_1' + C_2 \partial/\partial X') z_2^{(0)} = 0, \end{aligned}\right\} \tag{47}$$

whence we may look for solutions

$$z_n^{(0)} = M_n(T_2') \exp[i\kappa(X' - C_2 T_1')] + N_n(T_2') \exp[-i\kappa(X' - C_2 T_1')]. \tag{48}$$

At the next order

$$\left.\begin{aligned} \left(\frac{\partial}{\partial T_2'} + i\lambda_1 \kappa^2 + 2\lambda_1 \Gamma \frac{\partial}{\partial X'}\right) z_1^{(0)} + \left(\frac{\partial}{\partial T_1'} + C_2 \frac{\partial}{\partial X'}\right) z_1^{(1)} &= -i\alpha A (z_2^{(0)} + z_2^{(0)*}), \\ \left(\frac{\partial}{\partial T_2'} + i\lambda_2 \kappa^2\right) z_2^{(0)} + \left(\frac{\partial}{\partial T_1'} + C_2 \frac{\partial}{\partial X'}\right) z_2^{(1)} &= -\beta \left(A \frac{\partial z_1^{(0)*}}{\partial X'} + A^* \frac{\partial z_1^{(0)}}{\partial X'}\right), \end{aligned}\right\} \tag{49}$$

where for simplicity, $\alpha = \alpha(\mathbf{k}_1, \mathbf{k}_1, \mathbf{k}_2)$ and $\beta = \beta(\mathbf{k}_2, \mathbf{k}_1^*, \mathbf{k}_1)$. Suppression of secularities in (49), using (48), yields the equations

$$\left.\begin{aligned} [d/dT_2' + i\lambda_1 \kappa(\kappa + 2\Gamma)] M_1 &= -i\alpha A (M_2 + N_2^*), \\ [d/dT_2' + i\lambda_1 \kappa(\kappa - 2\Gamma)] N_1 &= -i\alpha A (M_2^* + N_2), \\ [d/dT_2' + i\lambda_2 \kappa^2] M_2 &= -i\beta (A^* M_1 + A N_1^*), \\ [d/dT_2' + i\lambda_2 \kappa^2] N_2 &= i\beta (A M_1^* + A^* N_1). \end{aligned}\right\} \tag{50}$$

Equations (50) have solutions $(M_1, N_1^*, M_2, N_2^*) \propto \exp(-i\Omega T_2')$, where

$$[(\Omega - 2\kappa\lambda_1 \Gamma)^2 - \lambda_1^2 \kappa^4][\Omega^2 - \lambda_2^2 \kappa^4] - 4\alpha\beta\lambda_1 \kappa^3 \Omega |A|^2 = 0. \tag{51}$$

At this stage it is convenient to set $|A| \equiv 1$ (thus defining the precise value of ϵ) and to note that $\lambda_2 = 0$ for Rossby waves. Then (51) gives $\Omega = 0$ or

$$\Omega[(\Omega - 2\kappa\lambda_1 \Gamma)^2 - \lambda_1^2 \kappa^4] - 4\alpha\beta\lambda_1 \kappa^3 = 0. \tag{52}$$

This equation has one or three real roots, any complex roots occurring in conjugate pairs, so that there is at most one growing root with $\mathrm{Im}\,\Omega = 0$ for any value of the coefficients.

Since the analyses of triad and side-band interaction are valid for $|k - \eta_c l| > O(\epsilon)$ while that of this section requires $|k - \eta_c l| \leqslant O(\epsilon^{\frac{2}{3}})$, the ranges of applicability overlap and in the limit $|\Gamma| \to \infty$ (on a scale of order unity; for consistency, $|\Gamma| \ll \epsilon^{-\frac{2}{3}}$) we recover from (52) the solutions of the previous sections in the limit $|k - \eta_c l| \to 0$.

With $|\Gamma| \to \infty$ and $\kappa \leqslant O(1)$, (52) has the solutions

$$\Omega \simeq \alpha\beta\kappa / \lambda_1 \Gamma^2 \tag{53}$$

and

$$\Omega \simeq 2\lambda_1 \Gamma \pm [\lambda_1^2 \kappa^4 + 2\alpha\beta\kappa^2/\Gamma]. \tag{54}$$

Bearing in mind (45) and (46) and the fact that $\lambda_2 = 0$ it is apparent that, in this limit, the solution (53) (together with $\Omega = 0$) corresponds to the stable side-band mode (i) of §5 while (54) represents mode (ii), corresponding precisely to (39). The latter mode is unstable if $\alpha\beta/\Gamma < 0$, i.e. $k < \eta_c l$ (since $\alpha\beta > 0$).

The resonant-triad solutions are recovered in the limit $|\Gamma| \to \infty$, $|\kappa/\Gamma| = O(1)$, when $\mathrm{Im}\,\Omega = 0$ unless

$$\kappa^2 = 4\Gamma^2, \tag{55}$$

in which case

$$\Omega^2 \simeq -2\alpha\beta\Gamma; \tag{56}$$

instability occurs for $\Gamma > 0$, i.e. $k > \eta_c l$. By virtue of (48) we may, without loss of generality, define $\kappa > 0$, so that the appropriate root of (55) is $\kappa = 2\Gamma$, in agreement with (27). It follows from (50) that $|N_1|/|M_1| \propto |\Gamma|^{-\frac{1}{2}}$; thus the triad structure of §4 is recovered as $|\Gamma| \to \infty$. By expanding the interaction coefficients in (20) about the long-wave resonance, and using (55), we find with $k_1 = \eta_c l + \Gamma\epsilon^{\frac{2}{3}}$, $k_2 = -\kappa\epsilon^{\frac{2}{3}}$ and $k_3 = -\eta_c l - (\Gamma - \kappa)\epsilon^{\frac{2}{3}}$ that $\alpha(\mathbf{k}_2, \mathbf{k}_3^*, \mathbf{k}_1^*) \simeq 2\beta\Gamma\epsilon^{\frac{2}{3}}$ while $\alpha(\mathbf{k}_3, \mathbf{k}_1^*, \mathbf{k}_2^*) \simeq \alpha$. Noting that $(C_2 - C_3)^2 = O(\epsilon^{\frac{4}{3}})$ and that the first term on the right side of (20) is represented here in (48) on the time scale T_1, it is clear that (20) and (54) are exactly equivalent in the appropriate limits.

Between these limits, the evolution of the unstable solution is continuous. Using the relevant parameter values $\alpha = 1{\cdot}18$, $\beta = 0{\cdot}23$ and $\lambda_1 = 0{\cdot}55$, figure 3 shows the maximum growth rate $\mathrm{Im}\,\Omega$ of the unstable solution to (54) as a function of Γ and the value of κ to which this maximum corresponds. Also shown is the range of κ for which (54) has unstable roots. The wave is unstable for all Γ and the single unstable mode evolves continuously from the side-band solution (with $\Omega \to 0$ on this scaling) as $\Gamma \to -\infty$ to the triad instability ($\Omega \to \infty$) as $\Gamma \to \infty$. In view of the discussion at the end of §4 concerning the reason for the loss of validity of the triad analysis as $k - \eta_c l \to 0^+$, this continuity between four-wave and three-wave interactions is not as surprising as it may at first seem.

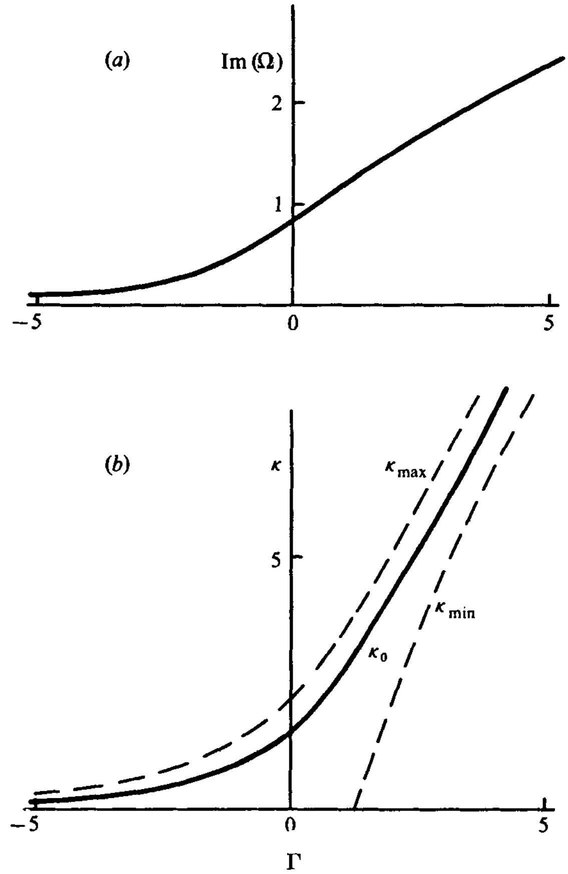

FIGURE 3. (a) Maximum growth rate Im Ω of the unstable solution to (52). (b) Solid curve, long-wave wavenumber κ_0 corresponding to the maximum growth rate in (a); dashed curves, range of κ for which instability occurs.

7. Summary and discussion

In an inviscid, zonally unbounded channel of fluid, a Rossby wave of gravest meridional structure is unstable to resonant-triad interactions if the ratio of its zonal and meridional wavenumbers k/l is greater than $\eta_c = 0{\cdot}681$. A wave of higher meridional structure is always unstable if its zonal wavenumber is non-zero. Zonally long waves ($k/l \leqslant \eta_c$) are unstable to side-band/long-wave perturbations; this process, being much slower than the triad interaction, will be of importance only in the case $l = 1$, when the wave would otherwise be stable. The transition at $k/l = \eta_c$ corresponds to a resonance in which the wave group velocity is equal to the long-wave speed. The growth rates in this regime are formally $O(\epsilon^{\frac{4}{3}})$ but the nature of the transition is one of continuous evolution from the side-band to the triad instability.

Instability may also occur via third-order, quartet resonances: unbounded Rossby waves can interact in such a way (Newell 1969). The time scale of this interaction is of order ϵ^{-2}, so that the conclusions of this paper would be affected only to the extent that quartet instabilities might occur with $k/l < \eta_c$, $l = 1$, with growth rates perhaps numerically larger than, but formally of the same order as, those associated with the side-band resonance.

The long-wave resonance has been analysed with sufficient generality to suggest more general qualitative applicability of the results. For internal wave motions, the significance of the long-wave resonance at $k/l = \eta_c = 0{\cdot}77$ and of unstable side-band resonances was demonstrated by Grimshaw (1975). The transition is, however, more important in the Rossby-wave case, when long waves are stable to more rapid interactions.

The transition at $k/l = \eta_c$ may have greater significance than that suggested by a stability analysis alone, in that the state to which an initially sinusoidal wave evolves as a result of the instability may be quite different in the two regimes. In the similar internal wave problem, the side-band interaction is governed by a nonlinear Schrödinger equation (Grimshaw 1975) which has (perhaps stable) envelope solutions to which the wave may evolve. Triad instabilities, on the other hand, will probably lead to a wave entirely losing its identity. At least one of the initially small amplitude, growing perturbations is itself unstable (having $l \geqslant 2$) and breakdown in this regime is likely to generate motion on all scales, as in the experiments of, for example, McEwan *et al.* (1972) and Martin *et al.* (1972).

In this context, it is noteworthy that Hide (1958; see also Hide & Mason 1975) found, over a wide range of l, that 'regular' baroclinic waves in a rotating annulus always have k/l less than about $1{\cdot}4$ (if viscosity is not too large); any change in the imposed conditions that would be expected to produce an increase in k leads to 'irregular' flow. Loesch (1974) studied the interaction between a weakly unstable wave and two neutral modes in a two-layer baroclinic model. The wavenumbers of the triad he considered (figure 16 of his paper) vary with k_1 in a manner similar to that of mode 1 in figure 1; in particular, there is a transition where, in the notation of this paper, $k/l \simeq 0{\cdot}7$, which may be shown to correspond to the long-wave resonance. This suggests that a more detailed study of baroclinic wave interactions may be an important step in the study of the evolution of such systems.

The analysis throughout this paper is based on the assumption of infinite channel length, i.e. a continuous zonal spectrum. If the channel is bounded zonally, the consequent spectral quantization may stabilize the interaction in all three regimes. Instability via the resonant-triad interaction requires that the perturbation wavenumber differs from the resonant value by less than a quantity $O(\epsilon)$ (i.e. in the notation of § 3, $|K| < \Delta_{123}$). If the channel length is $\alpha^{-1}L$ ($\alpha \ll 1$) the increment between allowable zonal wavenumbers is of order α; if $\alpha \leqslant O(\epsilon)$ then at least one of these allowable values will differ from resonance by less than the resonance bandwidth, whereas if $\alpha \gg \epsilon$ resonance is unlikely. The restrictions imposed by finite size are discussed in more detail and in greater generality in another paper (Plumb 1976). Breakdown via side-band interaction or at the long-wave resonance takes the form of a long modulation developing on the basic wave; in the former case, the wavelength of this modulation must be larger than a quantity or order $\epsilon^{-1}L$ [see (41)], so that similar constraints are imposed on both triad and side-band instability. At the long-wave resonance, however, the modulation wavelength is of order unity on a scale $\epsilon^{-\frac{2}{3}}$; hence in this special case, instability may occur when the channel length is of the order of $\epsilon^{-\frac{2}{3}}L$, a less severe requirement.

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